CHAPTER 4

MODELS FOR STATIONARY TIME SERIES

This chapter discusses the basic concepts of a broad class of parametric time series models—the autoregressive moving average (ARMA) models. These models have assumed great importance in modeling real-world processes.

4.1 General Linear Processes

We will always let $\{Y_t\}$ denote the observed time series. From here on we will also let $\{e_t\}$ represent an unobserved white noise series, that is, a sequence of identically distributed, zero-mean, independent random variables. For much of our work, the assumption of independence could be replaced by the weaker assumption that the $\{e_t\}$ are uncorrelated random variables, but we will not pursue that slight generality.

A *general linear process*, $\{Y_t\}$, is one that can be represented as a weighted linear combination of present and past white noise terms as

$$Y_t = e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \cdots \tag{4.1.1}$$

If the right-hand side of this expression is truly an infinite series, then certain conditions must be placed on the ψ-weights for the right-hand side to be meaningful mathematically. For our purposes, it suffices to assume that

$$\sum_{i=1}^{\infty} \psi_i^2 < \infty \tag{4.1.2}$$

We should also note that since $\{e_t\}$ is unobservable, there is no loss in the generality of Equation (4.1.2) if we assume that the coefficient on e_t is 1; effectively, $\psi_0 = 1$.

An important nontrivial example to which we will return often is the case where the ψ's form an exponentially decaying sequence

$$\psi_j = \phi^j$$

where ϕ is a number strictly between −1 and +1. Then

$$Y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots$$

For this example,

$$E(Y_t) = E(e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots) = 0$$

so that $\{Y_t\}$ has a constant mean of zero. Also,

$$\begin{aligned} Var(Y_t) &= Var(e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots) \\ &= Var(e_t) + \phi^2 Var(e_{t-1}) + \phi^4 Var(e_{t-2}) + \cdots \\ &= \sigma_e^2(1 + \phi^2 + \phi^4 + \cdots) \\ &= \frac{\sigma_e^2}{1-\phi^2} \quad \text{(by summing a geometric series)} \end{aligned}$$

Furthermore,

$$\begin{aligned} Cov(Y_t, Y_{t-1}) &= Cov(e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots, e_{t-1} + \phi e_{t-2} + \phi^2 e_{t-3} + \cdots) \\ &= Cov(\phi e_{t-1}, e_{t-1}) + Cov(\phi^2 e_{t-2}, \phi e_{t-2}) + \cdots \\ &= \phi\sigma_e^2 + \phi^3\sigma_e^2 + \phi^5\sigma_e^2 + \cdots \\ &= \phi\sigma_e^2(1 + \phi^2 + \phi^4 + \cdots) \\ &= \frac{\phi\sigma_e^2}{1-\phi^2} \quad \text{(again summing a geometric series)} \end{aligned}$$

Thus

$$Corr(Y_t, Y_{t-1}) = \left[\frac{\phi\sigma_e^2}{1-\phi^2}\right] \Big/ \left[\frac{\sigma_e^2}{1-\phi^2}\right] = \phi$$

In a similar manner, we can find $Cov(Y_t, Y_{t-k}) = \dfrac{\phi^k\sigma_e^2}{1-\phi^2}$ and thus

$$Corr(Y_t, Y_{t-k}) = \phi^k \tag{4.1.3}$$

It is important to note that the process defined in this way is stationary—the autocovariance structure depends only on time lag and not on absolute time. For a general linear process, $Y_t = e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \cdots$, calculations similar to those done above yield the following results:

$$E(Y_t) = 0 \qquad \gamma_k = Cov(Y_t, Y_{t-k}) = \sigma_e^2 \sum_{i=0}^{\infty} \psi_i \psi_{i+k} \qquad k \ge 0 \tag{4.1.4}$$

with $\psi_0 = 1$. A process with a nonzero mean μ may be obtained by adding μ to the right-hand side of Equation (4.1.1). Since the mean does not affect the covariance properties of a process, we assume a zero mean until we begin fitting models to data.

4.2 Moving Average Processes

In the case where only a finite number of the ψ-weights are nonzero, we have what is called a moving average process. In this case, we change notation† somewhat and write

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q} \tag{4.2.1}$$

We call such a series a **moving average of order *q*** and abbreviate the name to MA(q). The terminology moving average arises from the fact that Y_t is obtained by applying the weights $1, -\theta_1, -\theta_2, \ldots, -\theta_q$ to the variables $e_t, e_{t-1}, e_{t-2}, \ldots, e_{t-q}$ and then moving the weights and applying them to $e_{t+1}, e_t, e_{t-1}, \ldots, e_{t-q+1}$ to obtain Y_{t+1} and so on. Moving average models were first considered by Slutsky (1927) and Wold (1938).

The First-Order Moving Average Process

We consider in detail the simple but nevertheless important moving average process of order 1, that is, the MA(1) series. Rather than specialize the formulas in Equation (4.1.4), it is instructive to rederive the results. The model is $Y_t = e_t - \theta e_{t-1}$. Since only one θ is involved, we drop the redundant subscript 1. Clearly $E(Y_t) = 0$ and $Var(Y_t) = \sigma_e^2(1 + \theta^2)$. Now

$$\begin{aligned} Cov(Y_t, Y_{t-1}) &= Cov(e_t - \theta e_{t-1}, e_{t-1} - \theta e_{t-2}) \\ &= Cov(-\theta e_{t-1}, e_{t-1}) = -\theta\sigma_e^2 \end{aligned}$$

and

$$\begin{aligned} Cov(Y_t, Y_{t-2}) &= Cov(e_t - \theta e_{t-1}, e_{t-2} - \theta e_{t-3}) \\ &= 0 \end{aligned}$$

since there are no e's with subscripts in common between Y_t and Y_{t-2}. Similarly, $Cov(Y_t, Y_{t-k}) = 0$ whenever $k \geq 2$; that is, *the process has no correlation beyond lag* 1. This fact will be important later when we need to choose suitable models for real data.

In summary, for an MA(1) model $Y_t = e_t - \theta e_{t-1}$,

$$\left.\begin{aligned} E(Y_t) &= 0 \\ \gamma_0 &= Var(Y_t) = \sigma_e^2(1 + \theta^2) \\ \gamma_1 &= -\theta\sigma_e^2 \\ \rho_1 &= (-\theta)/(1 + \theta^2) \\ \gamma_k &= \rho_k = 0 \quad \text{for } k \geq 2 \end{aligned}\right\} \tag{4.2.2}$$

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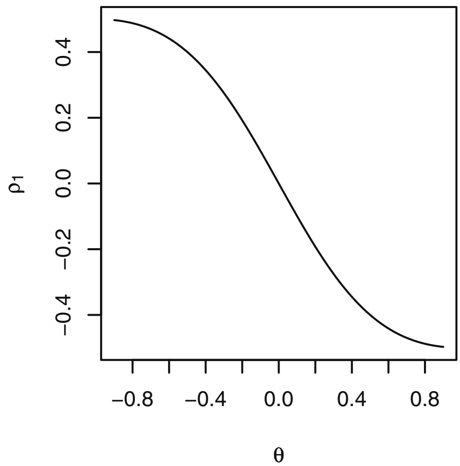
† The reason for this change will be evident later on. Some statistical software, for example R, uses plus signs before the thetas. Check with yours to see which convention it uses.

Some numerical values for ρ_1 versus θ in Equation (4.2.2) help illustrate the possibilities. Note that the ρ_1 values for negative θ can be obtained by simply negating the value given for the corresponding positive θ-value.

θ	$\rho_1 = -\theta/(1+\theta^2)$	θ	$\rho_1 = -\theta/(1+\theta^2)$
0.1	–0.099	0.6	–0.441
0.2	–0.192	0.7	–0.470
0.3	–0.275	0.8	–0.488
0.4	–0.345	0.9	–0.497
0.5	–0.400	1.0	–0.500

A calculus argument shows that the largest value that ρ_1 can attain is $\rho_1 = ½$ when $\theta = -1$ and the smallest value is $\rho_1 = -½$, which occurs when $\theta = +1$ (see Exercise 4.3). Exhibit 4.1 displays a graph of the lag 1 autocorrelation values for θ ranging from −1 to +1.

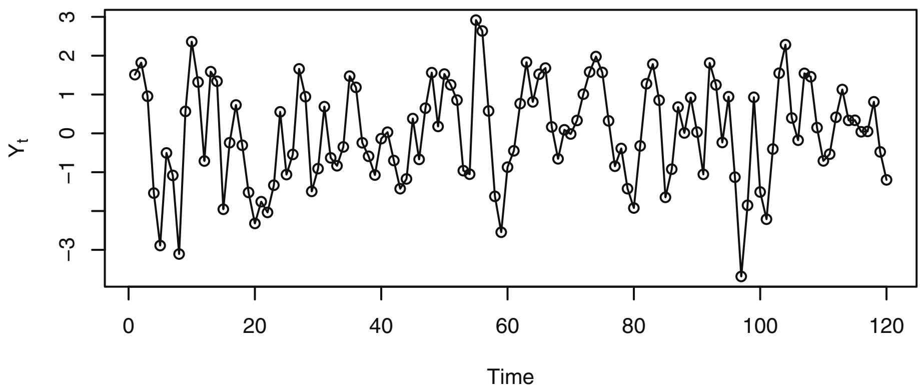
Exhibit 4.1 Lag 1 Autocorrelation of an MA(1) Process for Different θ



Exercise 4.4 asks you to show that when any nonzero value of θ is replaced by $1/\theta$, the *same* value for ρ_1 is obtained. For example, ρ_1 is the same for $\theta = ½$ as for $\theta = 1/(½) = 2$. If we knew that an MA(1) process had $\rho_1 = 0.4$, we still could not tell the precise value of θ. We will return to this troublesome point when we discuss *invertibility* in Section 4.5 on page 79.

Exhibit 4.2 shows a time plot of a simulated MA(1) series with $\theta = -0.9$ and normally distributed white noise. Recall from Exhibit 4.1 that $\rho_1 = 0.4972$ for this model; thus there is moderately strong positive correlation at lag 1. This correlation is evident in the plot of the series since consecutive observations tend to be closely related. If an observation is above the mean level of the series, then the next observation also tends to be above the mean. The plot is relatively smooth over time, with only occasional large fluctuations.

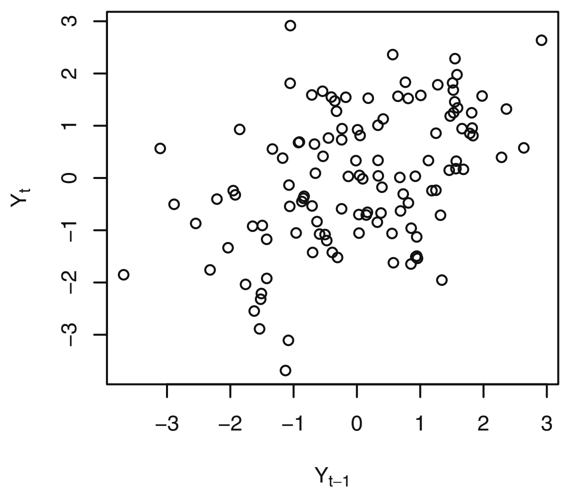
Exhibit 4.2 Time Plot of an MA(1) Process with θ = −0.9




```
> win.graph(width=4.875,height=3,pointsize=8)
> data(ma1.2.s); plot(ma1.2.s,ylab=expression(Y[t]),type='o')
```

The lag 1 autocorrelation is even more apparent in Exhibit 4.3, which plots Y_t versus Y_{t-1}. Note the moderately strong upward trend in this plot.

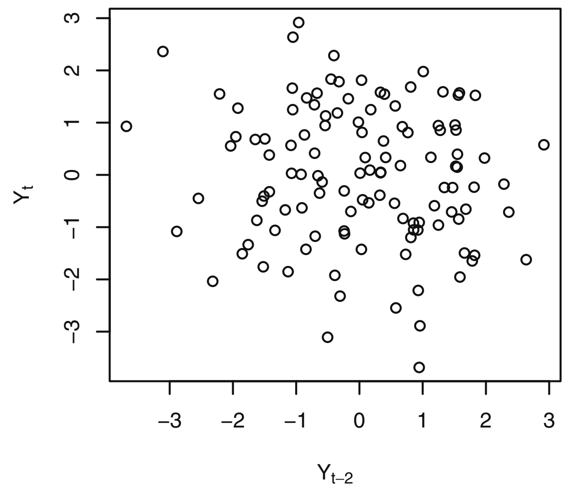
Exhibit 4.3 Plot of Y_t versus Y_{t-1} for MA(1) Series in Exhibit 4.2




```
> win.graph(width=3,height=3,pointsize=8)
> plot(y=ma1.2.s,x=zlag(ma1.2.s),ylab=expression(Y[t]),
   xlab=expression(Y[t-1]),type='p')
```

The plot of Y_t versus Y_{t-2} in Exhibit 4.4 gives a strong visualization of the zero autocorrelation at lag 2 for this model.

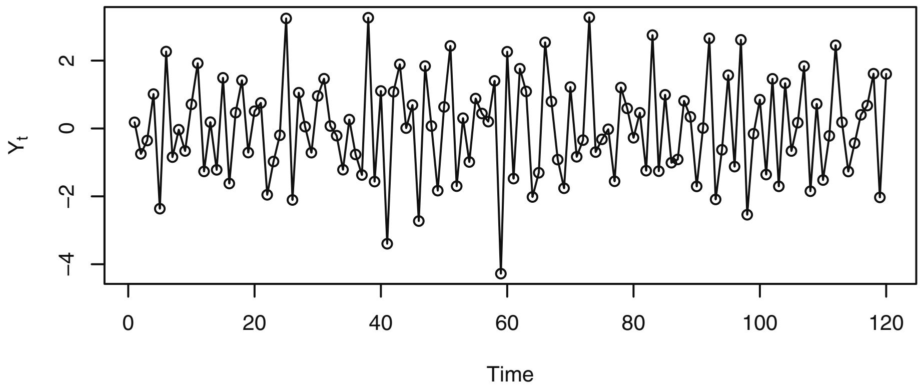
Exhibit 4.4 Plot of Y_t versus Y_{t-2} for MA(1) Series in Exhibit 4.2




```
> plot(y=ma1.2.s,x=zlag(ma1.2.s,2),ylab=expression(Y[t]),
  xlab=expression(Y[t-2]),type='p')
```

A somewhat different series is shown in Exhibit 4.5. This is a simulated MA(1) series with $\theta = +0.9$. Recall from Exhibit 4.1 that $\rho_1 = -0.497$ for this model; thus there is moderately strong negative correlation at lag 1. This correlation can be seen in the plot of the series since consecutive observations tend to be on opposite sides of the zero mean. If an observation is above the mean level of the series, then the next observation tends to be below the mean. The plot is quite jagged over time—especially when compared with the plot in Exhibit 4.2.

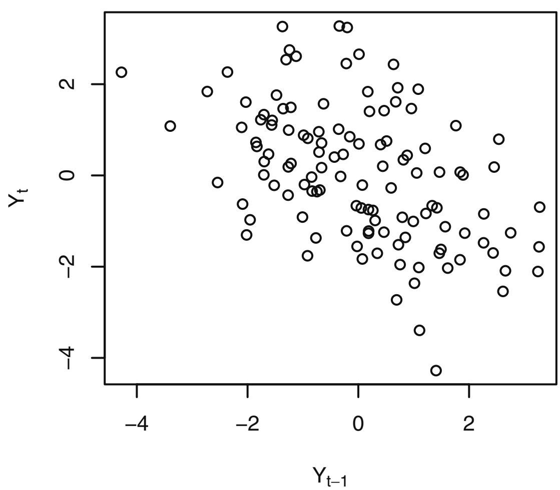
Exhibit 4.5 Time Plot of an MA(1) Process with θ = +0.9




```
> win.graph(width=4.875,height=3,pointsize=8)
> data(ma1.1.s)
> plot(ma1.1.s,ylab=expression(Y[t]),type='o')
```

The negative lag 1 autocorrelation is even more apparent in the lag plot of Exhibit 4.6.

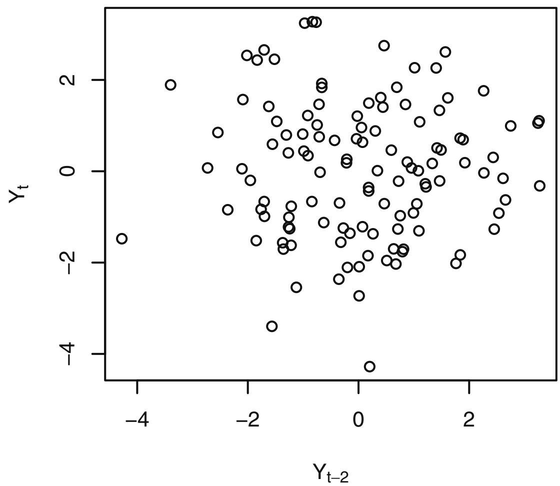
Exhibit 4.6 Plot of Y_t versus Y_{t-1} for MA(1) Series in Exhibit 4.5




```
> win.graph(width=3, height=3,pointsize=8)
> plot(y=ma1.1.s,x=zlag(ma1.1.s),ylab=expression(Y[t]),
   xlab=expression(Y[t-1]),type='p')
```

The plot of Y_t versus Y_{t-2} in Exhibit 4.7 displays the zero autocorrelation at lag 2 for this model.

Exhibit 4.7 Plot of Y_t versus Y_{t-2} for MA(1) Series in Exhibit 4.5




```
> plot(y=ma1.1.s,x=zlag(ma1.1.s,2),ylab=expression(Y[t]),
   xlab=expression(Y[t-2]),type='p')
```

MA(1) processes have no autocorrelation beyond lag 1, but by increasing the order of the process, we can obtain higher-order correlations.

The Second-Order Moving Average Process

Consider the moving average process of order 2:

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}$$

Here

$$\gamma_0 = Var(Y_t) = Var(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}) = (1 + \theta_1^2 + \theta_2^2)\sigma_e^2$$

$$\begin{aligned}
\gamma_1 &= Cov(Y_t, Y_{t-1}) = Cov(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}, e_{t-1} - \theta_1 e_{t-2} - \theta_2 e_{t-3}) \\
&= Cov(-\theta_1 e_{t-1}, e_{t-1}) + Cov(-\theta_1 e_{t-2}, -\theta_2 e_{t-2}) \\
&= [-\theta_1 + (-\theta_1)(-\theta_2)]\sigma_e^2 \\
&= (-\theta_1 + \theta_1\theta_2)\sigma_e^2
\end{aligned}$$

and

$$\begin{aligned}\gamma_2 &= Cov(Y_t, Y_{t-2}) = Cov(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}, e_{t-2} - \theta_1 e_{t-3} - \theta_2 e_{t-4})\\ &= Cov(-\theta_2 e_{t-2}, e_{t-2})\\ &= -\theta_2 \sigma_e^2\end{aligned}$$

Thus, for an MA(2) process,

$$\begin{aligned}\rho_1 &= \frac{-\theta_1 + \theta_1\theta_2}{1 + \theta_1^2 + \theta_2^2}\\ \rho_2 &= \frac{-\theta_2}{1 + \theta_1^2 + \theta_2^2}\\ \rho_k &= 0 \quad \text{for } k = 3, 4, \ldots\end{aligned} \tag{4.2.3}$$

For the specific case $Y_t = e_t - e_{t-1} + 0.6e_{t-2}$, we have

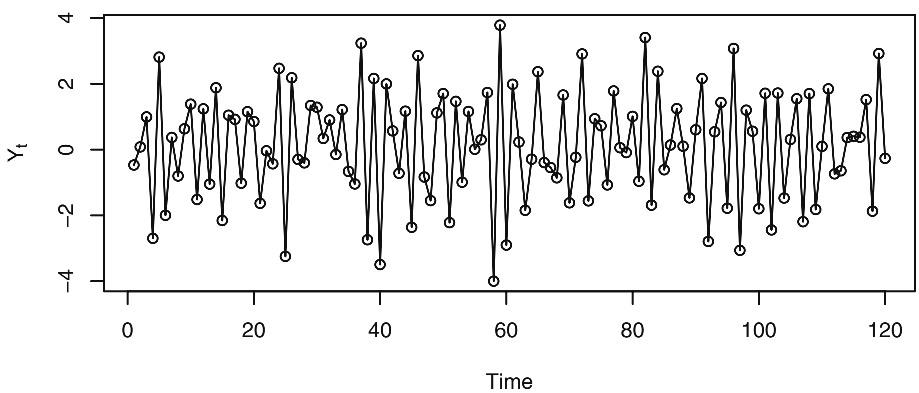
$$\rho_1 = \frac{-1 + (1)(-0.6)}{1 + (1)^2 + (-0.6)^2} = \frac{-1.6}{2.36} = -0.678$$

and

$$\rho_2 = \frac{0.6}{2.36} = 0.254$$

A time plot of a simulation of this MA(2) process is shown in Exhibit 4.8. The series tends to move back and forth across the mean in one time unit. This reflects the fairly strong negative autocorrelation at lag 1.

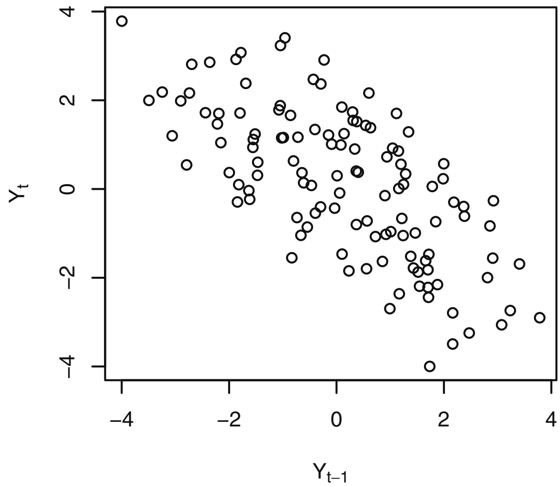
Exhibit 4.8 Time Plot of an MA(2) Process with $\theta_1 = 1$ and $\theta_2 = -0.6$




```
> win.graph(width=4.875, height=3,pointsize=8)
> data(ma2.s); plot(ma2.s,ylab=expression(Y[t]),type='o')
```

The plot in Exhibit 4.9 reflects that negative autocorrelation quite dramatically.

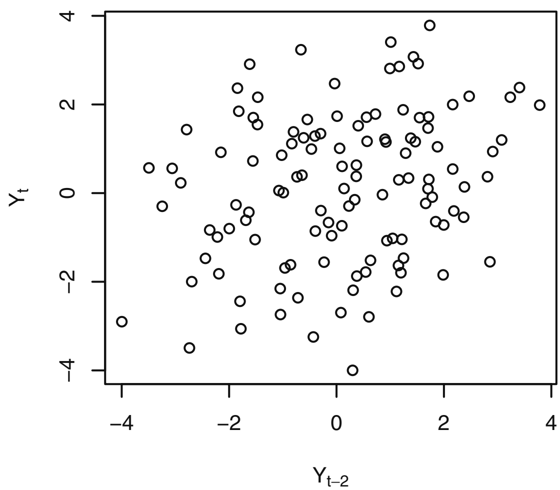
Exhibit 4.9 Plot of Y_t versus Y_{t-1} for MA(2) Series in Exhibit 4.8




```
> win.graph(width=3,height=3,pointsize=8)
> plot(y=ma2.s,x=zlag(ma2.s),ylab=expression(Y[t]),
   xlab=expression(Y[t-1]),type='p')
```

The weak positive autocorrelation at lag 2 is displayed in Exhibit 4.10.

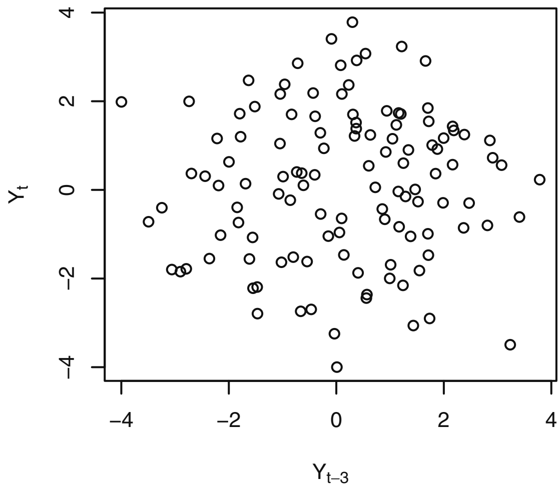
Exhibit 4.10 Plot of Y_t versus Y_{t-2} for MA(2) Series in Exhibit 4.8




```
> plot(y=ma2.s,x=zlag(ma2.s,2),ylab=expression(Y[t]),
   xlab=expression(Y[t-2]),type='p')
```

Finally, the lack of autocorrelation at lag 3 is apparent from the scatterplot in Exhibit 4.11.

Exhibit 4.11 Plot of Y_t versus Y_{t-3} for MA(2) Series in Exhibit 4.8




```
> plot(y=ma2.s,x=zlag(ma2.s,3),ylab=expression(Y[t]),
   xlab=expression(Y[t-3]),type='p')
```

The General MA(q) Process

For the general MA(q) process $Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}$, similar calculations show that

$$\gamma_0 = (1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2)\sigma_e^2 \tag{4.2.4}$$

and

$$\rho_k = \begin{cases} \dfrac{-\theta_k + \theta_1\theta_{k+1} + \theta_2\theta_{k+2} + \cdots + \theta_{q-k}\theta_q}{1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2} & \text{for } k = 1, 2, \ldots, q \\ 0 \qquad \text{for } k > q \end{cases} \tag{4.2.5}$$

where the numerator of ρ_q is just $-\theta_q$. The autocorrelation function "cuts off" after lag q; that is, it is zero. Its shape can be almost anything for the earlier lags. Another type of process, the autoregressive process, provides models for alternative autocorrelation patterns.

4.3 Autoregressive Processes

Autoregressive processes are as their name suggests—regressions on themselves. Specifically, a pth-order **autoregressive process** $\{Y_t\}$ satisfies the equation

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t \tag{4.3.1}$$

The current value of the series Y_t is a linear combination of the p most recent past values of itself plus an "innovation" term e_t that incorporates everything new in the series at time t that is not explained by the past values. Thus, for every t, we assume that e_t is independent of $Y_{t-1}, Y_{t-2}, Y_{t-3}, \ldots$. Yule (1926) carried out the original work on autoregressive processes.†

The First-Order Autoregressive Process

Again, it is instructive to consider the first-order model, abbreviated AR(1), in detail. Assume the series is stationary and satisfies

$$Y_t = \phi Y_{t-1} + e_t \tag{4.3.2}$$

where we have dropped the subscript 1 from the coefficient ϕ for simplicity. As usual, in these initial chapters, we assume that the process mean has been subtracted out so that the series mean is zero. The conditions for stationarity will be considered later.

We first take variances of both sides of Equation (4.3.2) and obtain

$$\gamma_0 = \phi^2 \gamma_0 + \sigma_e^2$$

Solving for γ_0 yields

$$\gamma_0 = \frac{\sigma_e^2}{1-\phi^2} \tag{4.3.3}$$

Notice the immediate implication that $\phi^2 < 1$ or that $|\phi| < 1$. Now take Equation (4.3.2), multiply both sides by Y_{t-k} ($k = 1, 2,\ldots$), and take expected values

$$E(Y_{t-k}Y_t) = \phi E(Y_{t-k}Y_{t-1}) + E(e_t Y_{t-k})$$

or

$$\gamma_k = \phi \gamma_{k-1} + E(e_t Y_{t-k})$$

Since the series is assumed to be stationary with zero mean, and since e_t is independent of Y_{t-k}, we obtain

$$E(e_t Y_{t-k}) = E(e_t)E(Y_{t-k}) = 0$$

and so

† Recall that we are assuming that Y_t has zero mean. We can always introduce a nonzero mean by replacing Y_t by $Y_t - \mu$ throughout our equations.

$$\gamma_k = \phi\gamma_{k-1} \qquad \text{for } k = 1, 2, 3, \ldots \tag{4.3.4}$$

Setting $k = 1$, we get $\gamma_1 = \phi\gamma_0 = \phi\sigma_e^2/(1-\phi^2)$. With $k = 2$, we obtain $\gamma_2 = \phi^2\sigma_e^2/(1-\phi^2)$. Now it is easy to see that in general

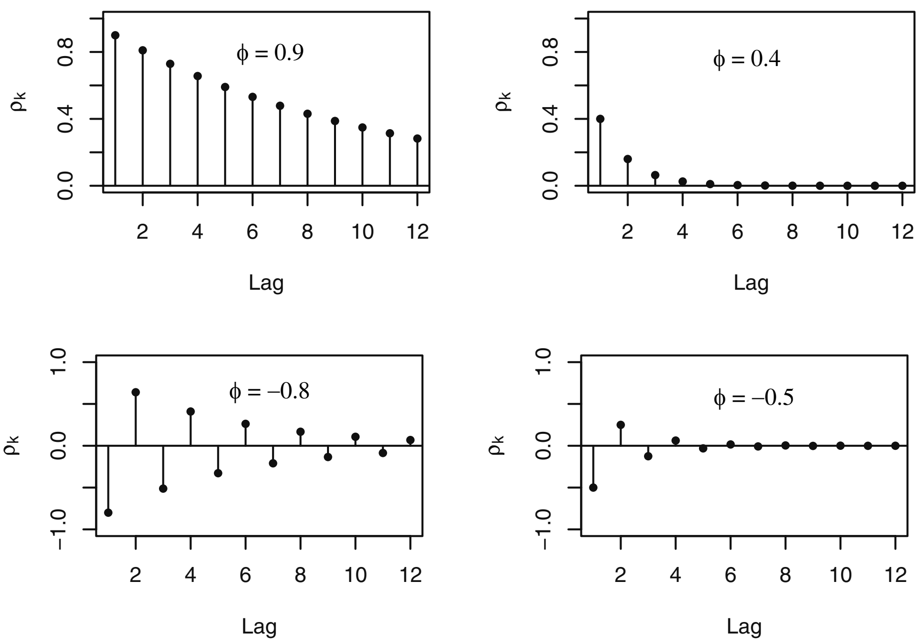
$$\gamma_k = \phi^k \frac{\sigma_e^2}{1-\phi^2} \tag{4.3.5}$$

and thus

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \phi^k \qquad \text{for } k = 1, 2, 3, \ldots \tag{4.3.6}$$

Since $|\phi| < 1$, the magnitude of the autocorrelation function decreases exponentially as the number of lags, k, increases. If $0 < \phi < 1$, all correlations are positive; if $-1 < \phi < 0$, the lag 1 autocorrelation is negative ($\rho_1 = \phi$) and the signs of successive autocorrelations alternate from positive to negative, with their magnitudes decreasing exponentially. Portions of the graphs of several autocorrelation functions are displayed in Exhibit 4.12.

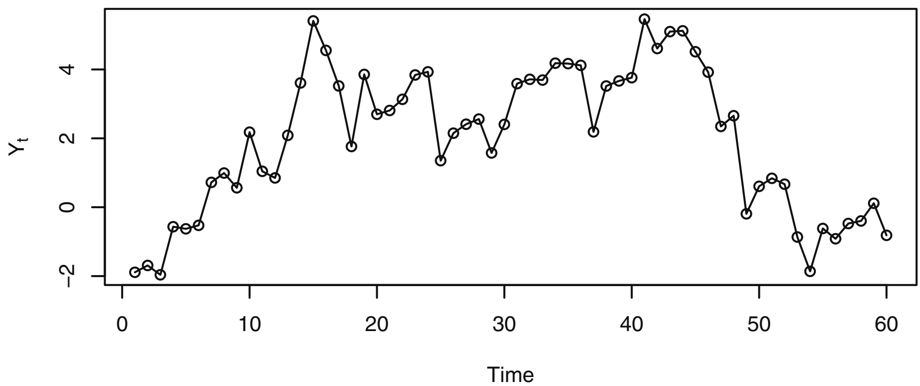
Exhibit 4.12 Autocorrelation Functions for Several AR(1) Models



Notice that for ϕ near ± 1, the exponential decay is quite slow (for example, $(0.9)^6 = 0.53$), but for smaller ϕ, the decay is quite rapid (for example, $(0.4)^6 = 0.00410$). With ϕ near ± 1, the strong correlation will extend over many lags and produce a relatively smooth series if ϕ is positive and a very jagged series if ϕ is negative.

Exhibit 4.13 displays the time plot of a simulated AR(1) process with $\phi = 0.9$. Notice how infrequently the series crosses its theoretical mean of zero. There is a lot of inertia in the series—it hangs together, remaining on the same side of the mean for extended periods. An observer might claim that the series has several trends. We know that in fact the theoretical mean is zero for all time points. The illusion of trends is due to the strong autocorrelation of neighboring values of the series.

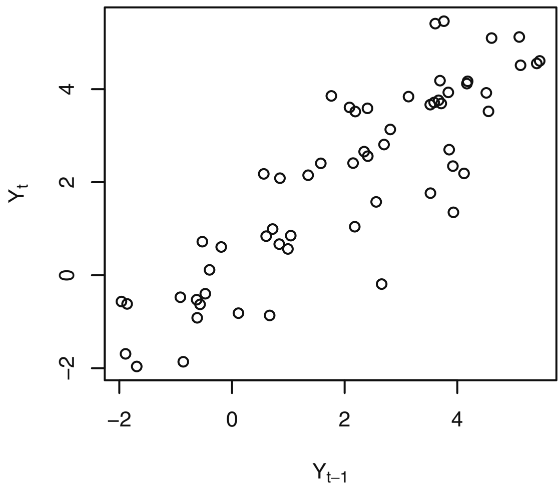
Exhibit 4.13 Time Plot of an AR(1) Series with ϕ = 0.9




```
> win.graph(width=4.875, height=3,pointsize=8)
> data(ar1.s); plot(ar1.s,ylab=expression(Y[t]),type='o')
```

The smoothness of the series and the strong autocorrelation at lag 1 are depicted in the lag plot shown in Exhibit 4.14.

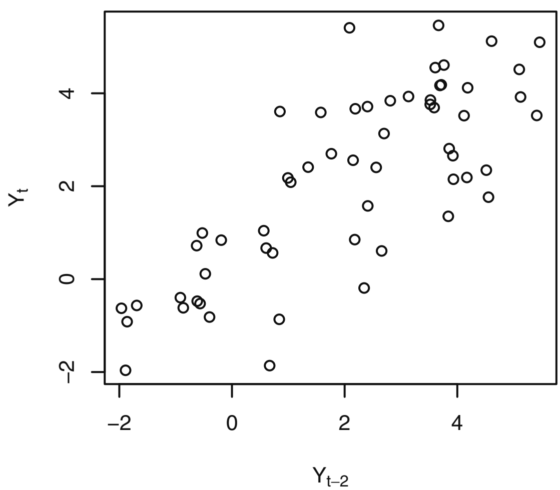
Exhibit 4.14 Plot of Y_t versus Y_{t-1} for AR(1) Series of Exhibit 4.13




```
> win.graph(width=3, height=3,pointsize=8)
> plot(y=ar1.s,x=zlag(ar1.s),ylab=expression(Y[t]),
   xlab=expression(Y[t-1]),type='p')
```

This AR(1) model also has strong positive autocorrelation at lag 2, namely $\rho_2 = (0.9)^2 = 0.81$. Exhibit 4.15 shows this quite well.

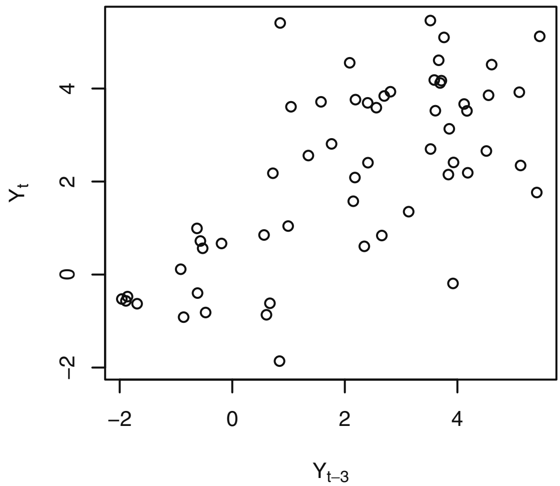
Exhibit 4.15 Plot of Y_t versus Y_{t-2} for AR(1) Series of Exhibit 4.13




```
> plot(y=ar1.s,x=zlag(ar1.s,2),ylab=expression(Y[t]),
   xlab=expression(Y[t-2]),type='p')
```

Finally, at lag 3, the autocorrelation is still quite high: $\rho_3 = (0.9)^3 = 0.729$. Exhibit 4.16 confirms this for this particular series.

Exhibit 4.16 Plot of Y_t versus Y_{t-3} for AR(1) Series of Exhibit 4.13




```
> plot(y=ar1.s,x=zlag(ar1.s,3),ylab=expression(Y[t]),
  xlab=expression(Y[t-3]),type='p')
```

The General Linear Process Version of the AR(1) Model

The recursive definition of the AR(1) process given in Equation (4.3.2) is extremely useful for interpretating the model. For other purposes, it is convenient to express the AR(1) model as a general linear process as in Equation (4.1.1). The recursive definition is valid for all t. If we use this equation with t replaced by $t-1$, we get $Y_{t-1} = \phi Y_{t-2} + e_{t-1}$. Substituting this into the original expression gives

$$\begin{aligned} Y_t &= \phi(\phi Y_{t-2} + e_{t-1}) + e_t \\ &= e_t + \phi e_{t-1} + \phi^2 Y_{t-2} \end{aligned}$$

If we repeat this substitution into the past, say $k - 1$ times, we get

$$Y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots + \phi^{k-1} e_{t-k+1} + \phi^k Y_{t-k} \tag{4.3.7}$$

Assuming $|\phi| < 1$ and letting k increase without bound, it seems reasonable (this is almost a rigorous proof) that we should obtain the infinite series representation

$$Y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \phi^3 e_{t-3} + \cdots \tag{4.3.8}$$

This is in the form of the general linear process of Equation (4.1.1) with $\psi_j = \phi^j$, which we already investigated in Section 4.1 on page 55. Note that this representation reemphasizes the need for the restriction $|\phi| < 1$.

Stationarity of an AR(1) Process

It can be shown that, subject to the restriction that e_t be independent of Y_{t-1}, Y_{t-2}, Y_{t-3},… and that $\sigma_e^2 > 0$, the solution of the AR(1) defining recursion $Y_t = \phi Y_{t-1} + e_t$ will be stationary if and only if $|\phi| < 1$. The requirement $|\phi| < 1$ is usually called the **stationarity condition** for the AR(1) process (See Box, Jenkins, and Reinsel, 1994, p. 54; Nelson, 1973, p. 39; and Wei, 2005, p. 32) even though more than stationarity is involved. See especially Exercises 4.16, 4.18, and 4.25.

At this point, we should note that the autocorrelation function for the AR(1) process has been derived in two different ways. The first method used the general linear process representation leading up to Equation (4.1.3). The second method used the defining recursion $Y_t = \phi Y_{t-1} + e_t$ and the development of Equations (4.3.4), (4.3.5), and (4.3.6). A third derivation is obtained by multiplying both sides of Equation (4.3.7) by Y_{t-k}, taking expected values of both sides, and using the fact that e_t, e_{t-1}, e_{t-2},..., $e_{t-(k-1)}$ are independent of Y_{t-k}. The second method should be especially noted since it will generalize nicely to higher-order processes.

The Second-Order Autoregressive Process

Now consider the series satisfying

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t \tag{4.3.9}$$

where, as usual, we assume that e_t is independent of Y_{t-1}, Y_{t-2}, Y_{t-3},… . To discuss stationarity, we introduce the **AR characteristic polynomial**

$$\phi(x) = 1 - \phi_1 x - \phi_2 x^2$$

and the corresponding **AR characteristic equation**

$$1 - \phi_1 x - \phi_2 x^2 = 0$$

We recall that a quadratic equation always has two roots (possibly complex).

Stationarity of the AR(2) Process

It may be shown that, subject to the condition that e_t is independent of Y_{t-1}, Y_{t-2}, Y_{t-3},..., a stationary solution to Equation (4.3.9) exists if and only if the roots of the AR characteristic equation exceed 1 in absolute value (modulus). We sometimes say that the roots should lie outside the unit circle in the complex plane. This statement will generalize to the *p*th-order case without change.†

---

† It also applies in the first-order case, where the AR characteristic equation is just $1 - \phi x = 0$ with root $1/\phi$, which exceeds 1 in absolute value if and only if $|\phi| < 1$.

In the second-order case, the roots of the quadratic characteristic equation are easily found to be

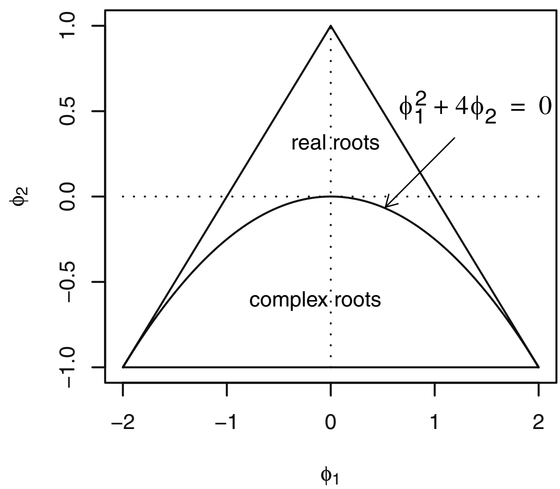
$$\frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_2} \tag{4.3.10}$$

For stationarity, we require that these roots exceed 1 in absolute value. In Appendix B, page 84, we show that this will be true if and only if three conditions are satisfied:

$$\phi_1 + \phi_2 < 1, \qquad \phi_2 - \phi_1 < 1, \qquad \text{and} \qquad |\phi_2| < 1 \tag{4.3.11}$$

As with the AR(1) model, we call these the **stationarity conditions** for the AR(2) model. This stationarity region is displayed in Exhibit 4.17.

Exhibit 4.17 Stationarity Parameter Region for AR(2) Process



The Autocorrelation Function for the AR(2) Process

To derive the autocorrelation function for the AR(2) case, we take the defining recursive relationship of Equation (4.3.9), multiply both sides by Y_{t-k}, and take expectations. Assuming stationarity, zero means, and that e_t is independent of Y_{t-k}, we get

$$\gamma_k = \phi_1\gamma_{k-1} + \phi_2\gamma_{k-2} \qquad \text{for } k = 1, 2, 3, \ldots \tag{4.3.12}$$

or, dividing through by γ_0,

$$\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2} \qquad \text{for } k = 1, 2, 3, \ldots \tag{4.3.13}$$

Equations (4.3.12) and/or (4.3.13) are usually called the **Yule-Walker equations**, especially the set of two equations obtained for $k = 1$ and 2. Setting $k = 1$ and using $\rho_0 = 1$ and $\rho_{-1} = \rho_1$, we get $\rho_1 = \phi_1 + \phi_2\rho_1$ and so

$$\rho_1 = \frac{\phi_1}{1-\phi_2} \tag{4.3.14}$$

Using the now known values for ρ_1 (and ρ_0), Equation (4.3.13) can be used with $k = 2$ to obtain

$$\begin{aligned}\rho_2 &= \phi_1\rho_1 + \phi_2\rho_0 \\ &= \frac{\phi_2(1-\phi_2)+\phi_1^2}{1-\phi_2}\end{aligned} \tag{4.3.15}$$

Successive values of ρ_k may be easily calculated numerically from the recursive relationship of Equation (4.3.13).

Although Equation (4.3.13) is very efficient for calculating autocorrelation values numerically from given values of ϕ_1 and ϕ_2, for other purposes it is desirable to have a more explicit formula for ρ_k. The form of the explicit solution depends critically on the roots of the characteristic equation $1 - \phi_1 x - \phi_2 x^2 = 0$. Denoting the reciprocals of these roots by G_1 and G_2, it is shown in Appendix B, page 84, that

$$G_1 = \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2} \qquad \text{and} \qquad G_2 = \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2}$$

For the case $G_1 \neq G_2$, it can be shown that we have

$$\rho_k = \frac{(1-G_2^2)G_1^{k+1} - (1-G_1^2)G_2^{k+1}}{(G_1-G_2)(1+G_1G_2)} \qquad \text{for } k \geq 0 \tag{4.3.16}$$

If the roots are complex (that is, if $\phi_1^2 + 4\phi_2 < 0$), then ρ_k may be rewritten as

$$\rho_k = R^k \frac{\sin(\Theta k + \Phi)}{\sin(\Phi)} \qquad \text{for } k \geq 0 \tag{4.3.17}$$

where $R = \sqrt{-\phi_2}$ and Θ and Φ are defined by $\cos(\Theta) = \phi_1/(2\sqrt{-\phi_2})$ and $\tan(\Phi) = [(1-\phi_2)/(1+\phi_2)]$.

For completeness, we note that if the roots are equal ($\phi_1^2 + 4\phi_2 = 0$), then we have

$$\rho_k = \left(1 + \frac{1+\phi_2}{1-\phi_2}k\right)\left(\frac{\phi_1}{2}\right)^k \qquad \text{for } k = 0, 1, 2, \ldots \tag{4.3.18}$$

A good discussion of the derivations of these formulas can be found in Fuller (1996, Section 2.5).

The specific details of these formulas are of little importance to us. We need only note that the autocorrelation function can assume a wide variety of shapes. In all cases, the magnitude of ρ_k dies out exponentially fast as the lag k increases. In the case of complex roots, ρ_k displays a damped sine wave behavior with **damping factor** R, $0 \leq R < 1$, **frequency** Θ, and **phase** Φ. Illustrations of the possible shapes are given in Exhibit 4.18. (The R function `ARMAacf` discussed on page 450 is useful for plotting.)

Exhibit 4.18 Autocorrelation Functions for Several AR(2) Models

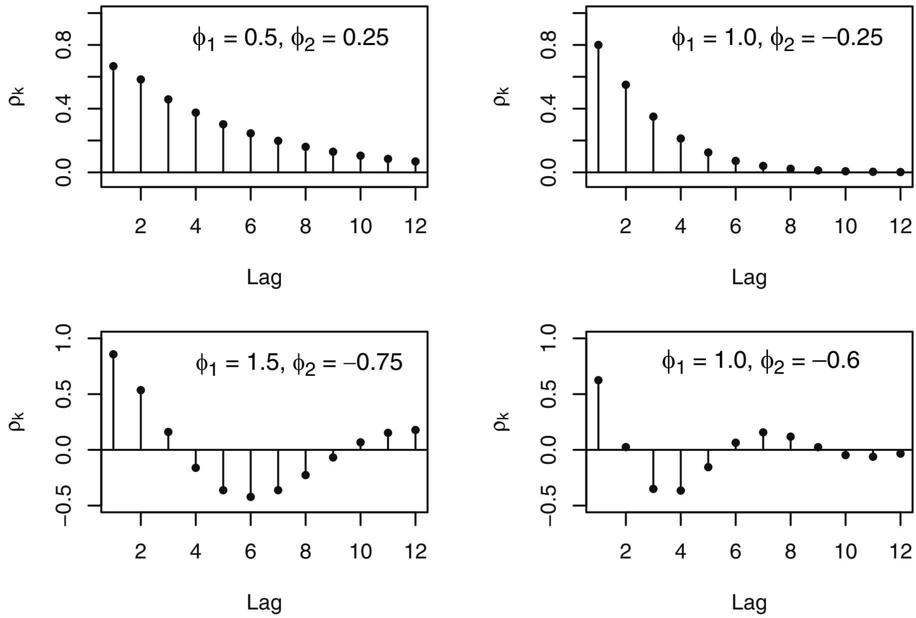
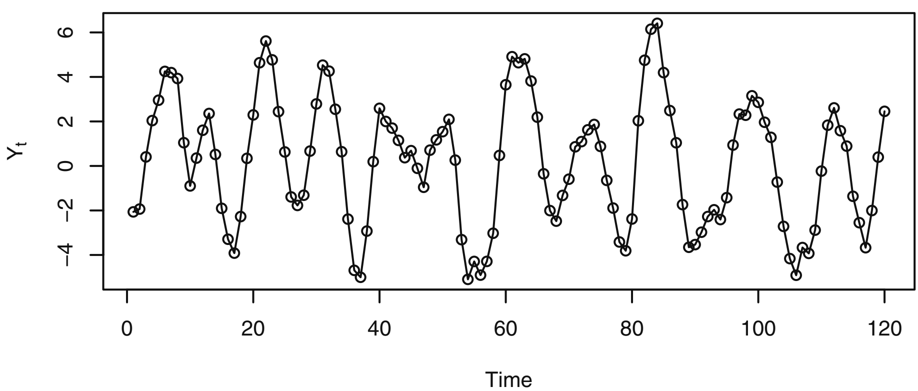


Exhibit 4.19 displays the time plot of a simulated AR(2) series with $\phi_1 = 1.5$ and $\phi_2 = -0.75$. The periodic behavior of ρ_k shown in Exhibit 4.18 is clearly reflected in the nearly periodic behavior of the series with the same period of 360/30 = 12 time units. If Θ is measured in radians, $2\pi/\Theta$ is sometimes called the **quasi-period** of the AR(2) process.

Exhibit 4.19 Time Plot of an AR(2) Series with ϕ_1 = 1.5 and ϕ_2 = −0.75




```
> win.graph(width=4.875,height=3,pointsize=8)
> data(ar2.s); plot(ar2.s,ylab=expression(Y[t]),type='o')
```

The Variance for the AR(2) Model

The process variance γ_0 can be expressed in terms of the model parameters ϕ_1, ϕ_2, and σ_e^2 as follows: Taking the variance of both sides of Equation (4.3.9) yields

$$\gamma_0 = (\phi_1^2 + \phi_2^2)\gamma_0 + 2\phi_1\phi_2\gamma_1 + \sigma_e^2 \tag{4.3.19}$$

Setting $k = 1$ in Equation (4.3.12) gives a second linear equation for γ_0 and γ_1, $\gamma_1 = \phi_1\gamma_0 + \phi_2\gamma_1$, which can be solved simultaneously with Equation (4.3.19) to obtain

$$\begin{aligned}\gamma_0 &= \frac{(1-\phi_2)\sigma_e^2}{(1-\phi_2)(1-\phi_1^2-\phi_2^2)-2\phi_2\phi_1^2} \\ &= \left(\frac{1-\phi_2}{1+\phi_2}\right)\frac{\sigma_e^2}{(1-\phi_2)^2-\phi_1^2}\end{aligned} \tag{4.3.20}$$

The ψ-Coefficients for the AR(2) Model

The ψ-coefficients in the general linear process representation for an AR(2) series are more complex than for the AR(1) case. However, we can substitute the general linear process representation using Equation (4.1.1) for Y_t, for Y_{t-1}, and for Y_{t-2} into $Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$. If we then equate coefficients of e_j, we get the recursive relationships

$$\left.\begin{aligned}\psi_0 &= 1 \\ \psi_1 - \phi_1\psi_0 &= 0 \\ \psi_j - \phi_1\psi_{j-1} - \phi_2\psi_{j-2} &= 0 \quad \text{for } j = 2, 3, \ldots\end{aligned}\right\} \tag{4.3.21}$$

These may be solved recursively to obtain $\psi_0 = 1$, $\psi_1 = \phi_1$, $\psi_2 = \phi_1^2 + \phi_2$, and so on. These relationships provide excellent numerical solutions for the ψ-coefficients for given numerical values of ϕ_1 and ϕ_2.

One can also show that, for $G_1 \neq G_2$, an explicit solution is

$$\psi_j = \frac{G_1^{j+1} - G_2^{j+1}}{G_1 - G_2} \tag{4.3.22}$$

where, as before, G_1 and G_2 are the reciprocals of the roots of the AR characteristic equation. If the roots are complex, Equation (4.3.22) may be rewritten as

$$\psi_j = R^j\left\{\frac{\sin[(j+1)\Theta]}{\sin(\Theta)}\right\} \tag{4.3.23}$$

a damped sine wave with the same damping factor R and frequency Θ as in Equation (4.3.17) for the autocorrelation function.

For completeness, we note that if the roots are equal, then

$$\psi_j = (1+j)\phi_1^j \tag{4.3.24}$$

The General Autoregressive Process

Consider now the pth-order autoregressive model

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t \tag{4.3.25}$$

with AR characteristic polynomial

$$\phi(x) = 1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p \tag{4.3.26}$$

and corresponding AR characteristic equation

$$1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p = 0 \tag{4.3.27}$$

As noted earlier, assuming that e_t is independent of Y_{t-1}, Y_{t-2}, Y_{t-3},... a stationary solution to Equation (4.3.27) exists if and only if the p roots of the AR characteristic equation each exceed 1 in absolute value (modulus). Other relationships between polynomial roots and coefficients may be used to show that the following two inequalities are necessary for stationarity. That is, for the roots to be greater than 1 in modulus, it is necessary, but not sufficient, that both

$$\left.\begin{array}{r} \phi_1 + \phi_2 + \cdots + \phi_p < 1 \\ \text{and} \qquad |\phi_p| < 1 \end{array}\right\} \tag{4.3.28}$$

Assuming stationarity and zero means, we may multiply Equation (4.3.25) by Y_{t-k}, take expectations, divide by γ_0, and obtain the important recursive relationship

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} + \phi_3 \rho_{k-3} + \cdots + \phi_p \rho_{k-p} \qquad \text{for } k \ge 1 \tag{4.3.29}$$

Putting $k = 1, 2, ...,$ and p into Equation (4.3.29) and using $\rho_0 = 1$ and $\rho_{-k} = \rho_k$, we get the general **Yule-Walker equations**

$$\left.\begin{array}{rcl} \rho_1 &=& \phi_1 + \phi_2 \rho_1 + \phi_3 \rho_2 + \cdots + \phi_p \rho_{p-1} \\ \rho_2 &=& \phi_1 \rho_1 + \phi_2 + \phi_3 \rho_1 + \cdots + \phi_p \rho_{p-2} \\ \vdots && \\ \rho_p &=& \phi_1 \rho_{p-1} + \phi_2 \rho_{p-2} + \phi_3 \rho_{p-3} + \cdots + \phi_p \end{array}\right\} \tag{4.3.30}$$

Given numerical values for $\phi_1, \phi_2, ..., \phi_p$, these linear equations can be solved to obtain numerical values for $\rho_1, \rho_2, ..., \rho_p$. Then Equation (4.3.29) can be used to obtain numerical values for ρ_k at any number of higher lags.

Noting that

$$E(e_t Y_t) = E[e_t(\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t)] = E(e_t^2) = \sigma_e^2$$

we may multiply Equation (4.3.25) by Y_t, take expectations, and find

$$\gamma_0 = \phi_1 \gamma_1 + \phi_2 \gamma_2 + \cdots + \phi_p \gamma_p + \sigma_e^2$$

which, using $\rho_k = \gamma_k/\gamma_0$, can be written as

$$\gamma_0 = \frac{\sigma_e^2}{1 - \phi_1\rho_1 - \phi_2\rho_2 - \cdots - \phi_p\rho_p} \tag{4.3.31}$$

and express the process variance γ_0 in terms of the parameters σ_e^2, $\phi_1, \phi_2, \ldots, \phi_p$, and the now known values of $\rho_1, \rho_2, \ldots, \rho_p$. Of course, explicit solutions for ρ_k are essentially impossible in this generality, but we can say that ρ_k will be a linear combination of exponentially decaying terms (corresponding to the real roots of the characteristic equation) and damped sine wave terms (corresponding to the complex roots of the characteristic equation).

Assuming stationarity, the process can also be expressed in the general linear process form of Equation (4.1.1), but the ψ-coefficients are complicated functions of the parameters $\phi_1, \phi_2, \ldots, \phi_p$. The coefficients can be found numerically; see Appendix C on page 85.

4.4 The Mixed Autoregressive Moving Average Model

If we assume that the series is partly autoregressive and partly moving average, we obtain a quite general time series model. In general, if

$$\begin{aligned} Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} \\ - \cdots - \theta_q e_{t-q} \end{aligned} \tag{4.4.1}$$

we say that $\{Y_t\}$ is a mixed **autoregressive moving average** process of orders p and q, respectively; we abbreviate the name to ARMA(p,q). As usual, we discuss an important special case first.†

The ARMA(1,1) Model

The defining equation can be written

$$Y_t = \phi Y_{t-1} + e_t - \theta e_{t-1} \tag{4.4.2}$$

To derive Yule-Walker type equations, we first note that

$$\begin{aligned} E(e_t Y_t) &= E[e_t(\phi Y_{t-1} + e_t - \theta e_{t-1})] \\ &= \sigma_e^2 \end{aligned}$$

and

† In mixed models, we assume that there are no common factors in the autoregressive and moving average polynomials. If there were, we could cancel them and the model would reduce to an ARMA model of lower order. For ARMA(1,1), this means $\theta \neq \phi$.

$$\begin{aligned} E(e_{t-1}Y_t) &= E[e_{t-1}(\phi Y_{t-1} + e_t - \theta e_{t-1})] \\ &= \phi\sigma_e^2 - \theta\sigma_e^2 \\ &= (\phi - \theta)\sigma_e^2 \end{aligned}$$

If we multiply Equation (4.4.2) by Y_{t-k} and take expectations, we have

$$\left.\begin{aligned} \gamma_0 &= \phi\gamma_1 + [1 - \theta(\phi - \theta)]\sigma_e^2 \\ \gamma_1 &= \phi\gamma_0 - \theta\sigma_e^2 \\ \gamma_k &= \phi\gamma_{k-1} \quad \text{for } k \ge 2 \end{aligned}\right\} \tag{4.4.3}$$

Solving the first two equations yields

$$\gamma_0 = \frac{(1 - 2\phi\theta + \theta^2)}{1 - \phi^2}\sigma_e^2 \tag{4.4.4}$$

and solving the simple recursion gives

$$\rho_k = \frac{(1 - \theta\phi)(\phi - \theta)}{1 - 2\theta\phi + \theta^2}\phi^{k-1} \quad \text{for } k \ge 1 \tag{4.4.5}$$

Note that this autocorrelation function decays exponentially as the lag k increases. The damping factor is ϕ, but the decay starts from initial value ρ_1, which also depends on θ. This is in contrast to the AR(1) autocorrelation, which also decays with damping factor ϕ but always from initial value $\rho_0 = 1$. For example, if $\phi = 0.8$ and $\theta = 0.4$, then $\rho_1 = 0.523$, $\rho_2 = 0.418$, $\rho_3 = 0.335$, and so on. Several shapes for ρ_k are possible, depending on the sign of ρ_1 and the sign of ϕ.

The general linear process form of the model can be obtained in the same manner that led to Equation (4.3.8). We find

$$Y_t = e_t + (\phi - \theta)\sum_{j=1}^{\infty}\phi^{j-1}e_{t-j}\ , \tag{4.4.6}$$

that is,

$$\psi_j = (\phi - \theta)\phi^{j-1} \qquad \text{for } j \ge 1$$

We should now mention the obvious stationarity condition $|\phi| < 1$, or equivalently the root of the AR characteristic equation $1 - \phi x = 0$ must exceed unity in absolute value.

For the general ARMA(p,q) model, we state the following facts without proof: Subject to the condition that e_t is independent of $Y_{t-1}, Y_{t-2}, Y_{t-3},\ldots$, a stationary solution to Equation (4.4.1) exists if and only if all the roots of the AR characteristic equation $\phi(x) = 0$ exceed unity in modulus.

If the stationarity conditions are satisfied, then the model can also be written as a general linear process with ψ-coefficients determined from

$$\left.\begin{aligned}
\psi_0 &= 1 \\
\psi_1 &= -\theta_1 + \phi_1 \\
\psi_2 &= -\theta_2 + \phi_2 + \phi_1\psi_1 \\
&\;\vdots \\
\psi_j &= -\theta_j + \phi_p\psi_{j-p} + \phi_{p-1}\psi_{j-p+1} + \cdots + \phi_1\psi_{j-1}
\end{aligned}\right\} \tag{4.4.7}$$

where we take $\psi_j = 0$ for $j < 0$ and $\theta_j = 0$ for $j > q$.

Again assuming stationarity, the autocorrelation function can easily be shown to satisfy

$$\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2} + \cdots + \phi_p\rho_{k-p} \qquad \text{for } k > q \tag{4.4.8}$$

Similar equations can be developed for $k = 1, 2, 3,\ldots, q$ that involve $\theta_1, \theta_2,\ldots, \theta_q$. An algorithm suitable for numerical computation of the complete autocorrelation function is given in Appendix C on page 85. (This algorithm is implemented in the R function named `ARMAacf`.)

4.5 Invertibility

We have seen that for the MA(1) process we get exactly the same autocorrelation function if θ is replaced by $1/\theta$. In the exercises, we find a similar problem with nonuniqueness for the MA(2) model. This lack of uniqueness of MA models, given their autocorrelation functions, must be addressed before we try to infer the values of parameters from observed time series. It turns out that this nonuniqueness is related to the seemingly unrelated question stated next.

An autoregressive process can always be reexpressed as a general linear process through the ψ-coefficients so that an AR process may also be thought of as an infinite-order moving average process. However, for some purposes, the autoregressive representations are also convenient. Can a moving average model be reexpressed as an autoregression?

To fix ideas, consider an MA(1) model:

$$Y_t = e_t - \theta e_{t-1} \tag{4.5.1}$$

First rewriting this as $e_t = Y_t + \theta e_{t-1}$ and then replacing t by $t - 1$ and substituting for e_{t-1} above, we get

$$\begin{aligned}
e_t &= Y_t + \theta(Y_{t-1} + \theta e_{t-2}) \\
&= Y_t + \theta Y_{t-1} + \theta^2 e_{t-2}
\end{aligned}$$

If $|\theta| < 1$, we may continue this substitution "infinitely" into the past and obtain the expression [compare with Equations (4.3.7) and (4.3.8)]

$$e_t = Y_t + \theta Y_{t-1} + \theta^2 Y_{t-2} + \cdots$$

or

$$Y_t = (-\theta Y_{t-1} - \theta^2 Y_{t-2} - \theta^3 Y_{t-3} - \cdots) + e_t \tag{4.5.2}$$

If $|\theta| < 1$, we see that the MA(1) model can be inverted into an infinite-order autoregressive model. We say that the MA(1) model is invertible if and only if $|\theta| < 1$.

For a general MA(q) or ARMA(p,q) model, we define the **MA characteristic polynomial** as

$$\theta(x) = 1 - \theta_1 x - \theta_2 x^2 - \theta_3 x^3 - \cdots - \theta_q x^q \tag{4.5.3}$$

and the corresponding **MA characteristic equation**

$$1 - \theta_1 x - \theta_2 x^2 - \theta_3 x^3 - \cdots - \theta_q x^q = 0 \tag{4.5.4}$$

It can be shown that the MA(q) model is **invertible**; that is, there are coefficients π_j such that

$$Y_t = \pi_1 Y_{t-1} + \pi_2 Y_{t-2} + \pi_3 Y_{t-3} + \cdots + e_t \tag{4.5.5}$$

if and only if the roots of the MA characteristic equation exceed 1 in modulus. (Compare this with stationarity of an AR model.)

It may also be shown that there is only one set of parameter values that yield an invertible MA process with a given autocorrelation function. For example, $Y_t = e_t + 2e_{t-1}$ and $Y_t = e_t + \frac{1}{2}e_{t-1}$ both have the same autocorrelation function, but only the second one with root -2 is invertible. From here on, we will restrict our attention to the physically sensible class of invertible models.

For a general ARMA(p,q) model, we require both stationarity and invertibility.

4.6 Summary

This chapter introduces the simple but very useful autoregressive, moving average (ARMA) time series models. The basic statistical properties of these models were derived in particular for the important special cases of moving averages of orders 1 and 2 and autoregressive processes of orders 1 and 2. Stationarity and invertibility issues have been pursued for these cases. Properties of mixed ARMA models have also been investigated. You should be well-versed in the autocorrelation properties of these models and the various representations of the models.

Exercises

4.1 Use first principles to find the autocorrelation function for the stationary process defined by

$$Y_t = 5 + e_t - \tfrac{1}{2}e_{t-1} + \tfrac{1}{4}e_{t-2}$$

4.2 Sketch the autocorrelation functions for the following MA(2) models with parameters as specified:
(a) $\theta_1 = 0.5$ and $\theta_2 = 0.4$.
(b) $\theta_1 = 1.2$ and $\theta_2 = -0.7$.
(c) $\theta_1 = -1$ and $\theta_2 = -0.6$.

4.3 Verify that for an MA(1) process

$$\max_{-\infty < \theta < \infty} \rho_1 = 0.5 \qquad \text{and} \qquad \min_{-\infty < \theta < \infty} \rho_1 = -0.5$$

4.4 Show that when θ is replaced by $1/\theta$, the autocorrelation function for an MA(1) process does not change.

4.5 Calculate and sketch the autocorrelation functions for each of the following AR(1) models. Plot for sufficient lags that the autocorrelation function has nearly died out.
(a) $\phi_1 = 0.6$.
(b) $\phi_1 = -0.6$.
(c) $\phi_1 = 0.95$. (Do out to 20 lags.)
(d) $\phi_1 = 0.3$.

4.6 Suppose that $\{Y_t\}$ is an AR(1) process with $-1 < \phi < +1$.
(a) Find the autocovariance function for $W_t = \nabla Y_t = Y_t - Y_{t-1}$ in terms of ϕ and σ_e^2.
(b) In particular, show that $Var(W_t) = 2\sigma_e^2/(1+\phi)$.

4.7 Describe the important characteristics of the autocorrelation function for the following models: (a) MA(1), (b) MA(2), (c) AR(1), (d) AR(2), and (e) ARMA(1,1).

4.8 Let $\{Y_t\}$ be an AR(2) process of the special form $Y_t = \phi_2 Y_{t-2} + e_t$. Use first principles to find the range of values of ϕ_2 for which the process is stationary.

4.9 Use the recursive formula of Equation (4.3.13) to calculate and then sketch the autocorrelation functions for the following AR(2) models with parameters as specified. In each case, specify whether the roots of the characteristic equation are real or complex. If the roots are complex, find the damping factor, R, and frequency, Θ, for the corresponding autocorrelation function when expressed as in Equation (4.3.17), on page 73.
(a) $\phi_1 = 0.6$ and $\phi_2 = 0.3$.
(b) $\phi_1 = -0.4$ and $\phi_2 = 0.5$.
(c) $\phi_1 = 1.2$ and $\phi_2 = -0.7$.
(d) $\phi_1 = -1$ and $\phi_2 = -0.6$.
(e) $\phi_1 = 0.5$ and $\phi_2 = -0.9$.
(f) $\phi_1 = -0.5$ and $\phi_2 = -0.6$.

4.10 Sketch the autocorrelation functions for each of the following ARMA models:
(a) ARMA(1,1) with $\phi = 0.7$ and $\theta = 0.4$.
(b) ARMA(1,1) with $\phi = 0.7$ and $\theta = -0.4$.

4.11 For the ARMA(1,2) model $Y_t = 0.8Y_{t-1} + e_t + 0.7e_{t-1} + 0.6e_{t-2}$, show that
(a) $\rho_k = 0.8\rho_{k-1}$ for $k > 2$.
(b) $\rho_2 = 0.8\rho_1 + 0.6\sigma_e^2/\gamma_0$.

4.12 Consider two MA(2) processes, one with $\theta_1 = \theta_2 = 1/6$ and another with $\theta_1 = -1$ and $\theta_2 = 6$.
(a) Show that these processes have the same autocorrelation function.
(b) How do the roots of the corresponding characteristic polynomials compare?

4.13 Let $\{Y_t\}$ be a stationary process with $\rho_k = 0$ for $k > 1$. Show that we must have $|\rho_1| \leq ½$. (Hint: Consider $Var(Y_{n+1} + Y_n + \cdots + Y_1)$ and then $Var(Y_{n+1} - Y_n + Y_{n-1} - \cdots \pm Y_1)$. Use the fact that both of these must be nonnegative for *all* n.)

4.14 Suppose that $\{Y_t\}$ is a zero mean, stationary process with $|\rho_1| < 0.5$ and $\rho_k = 0$ for $k > 1$. Show that $\{Y_t\}$ must be representable as an MA(1) process. That is, show that there is a white noise sequence $\{e_t\}$ such that $Y_t = e_t - \theta e_{t-1}$, where ρ_1 is correct and e_t is uncorrelated with Y_{t-k} for $k > 0$. (Hint: Choose θ such that $|\theta| < 1$ and $\rho_1 = -\theta/(1 + \theta^2)$; then let $e_t = \sum_{j=0}^{\infty}\theta^j Y_{t-j}$. If we assume that $\{Y_t\}$ is a normal process, e_t will also be normal, and zero correlation is equivalent to independence.)

4.15 Consider the AR(1) model $Y_t = \phi Y_{t-1} + e_t$. Show that if $|\phi| = 1$ the process cannot be stationary. (Hint: Take variances of both sides.)

4.16 Consider the "nonstationary" AR(1) model $Y_t = 3Y_{t-1} + e_t$.
(a) Show that $Y_t = -\sum_{j=1}^{\infty}\left(\frac{1}{3}\right)^j e_{t+j}$ satisfies the AR(1) equation.
(b) Show that the process defined in part (a) is stationary.
(c) In what way is this solution unsatisfactory?

4.17 Consider a process that satisfies the AR(1) equation $Y_t = ½Y_{t-1} + e_t$.
(a) Show that $Y_t = 10(½)^t + e_t + ½e_{t-1} + (½)^2e_{t-2} + \cdots$ is a solution of the AR(1) equation.
(b) Is the solution given in part (a) stationary?

4.18 Consider a process that satisfies the zero-mean, "stationary" AR(1) equation $Y_t = \phi Y_{t-1} + e_t$ with $-1 < \phi < +1$. Let c be any nonzero constant, and define $W_t = Y_t + c\phi^t$.
(a) Show that $E(W_t) = c\phi^t$.
(b) Show that $\{W_t\}$ satisfies the "stationary" AR(1) equation $W_t = \phi W_{t-1} + e_t$.
(c) Is $\{W_t\}$ stationary?

4.19 Consider an MA(6) model with $\theta_1 = 0.5$, $\theta_2 = -0.25$, $\theta_3 = 0.125$, $\theta_4 = -0.0625$, $\theta_5 = 0.03125$, and $\theta_6 = -0.015625$. Find a much simpler model that has nearly the same ψ-weights.

4.20 Consider an MA(7) model with $\theta_1 = 1$, $\theta_2 = -0.5$, $\theta_3 = 0.25$, $\theta_4 = -0.125$, $\theta_5 = 0.0625$, $\theta_6 = -0.03125$, and $\theta_7 = 0.015625$. Find a much simpler model that has nearly the same ψ-weights.

4.21 Consider the model $Y_t = e_{t-1} - e_{t-2} + 0.5e_{t-3}$.

(a) Find the autocovariance function for this process.

(b) Show that this is a certain ARMA(p,q) process in disguise. That is, identify values for p and q and for the θ's and φ's such that the ARMA(p,q) process has the same statistical properties as $\{Y_t\}$.

4.22 Show that the statement "The roots of $1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p = 0$ are greater than 1 in absolute value" is equivalent to the statement "The roots of $x^p - \phi_1 x^{p-1} - \phi_2 x^{p-2} - \cdots - \phi_p = 0$ are less than 1 in absolute value." (Hint: If G is a root of one equation, is $1/G$ a root of the other?)

4.23 Suppose that $\{Y_t\}$ is an AR(1) process with $\rho_1 = \phi$. Define the sequence $\{b_t\}$ as $b_t = Y_t - \phi Y_{t+1}$.

(a) Show that $Cov(b_t, b_{t-k}) = 0$ for all t and k.

(b) Show that $Cov(b_t, Y_{t+k}) = 0$ for all t and $k > 0$.

4.24 Let $\{e_t\}$ be a zero-mean, unit-variance white noise process. Consider a process that begins at time $t = 0$ and is defined recursively as follows. Let $Y_0 = c_1 e_0$ and $Y_1 = c_2 Y_0 + e_1$. Then let $Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$ for $t > 1$ as in an AR(2) process.

(a) Show that the process mean is zero.

(b) For particular values of ϕ_1 and ϕ_2 within the stationarity region for an AR(2) model, show how to choose c_1 and c_2 so that both $Var(Y_0) = Var(Y_1)$ and the lag 1 autocorrelation between Y_1 and Y_0 match that of a stationary AR(2) process with parameters ϕ_1 and ϕ_2.

(c) Once the process $\{Y_t\}$ is generated, show how to transform it to a new process that has any desired mean and variance. (This exercise suggests a convenient method for simulating stationary AR(2) processes.)

4.25 Consider an "AR(1)" process satisfying $Y_t = \phi Y_{t-1} + e_t$, where φ can be ***any*** number and $\{e_t\}$ is a white noise process such that e_t is independent of the past $\{Y_{t-1}, Y_{t-2}, \ldots\}$. Let Y_0 be a random variable with mean μ_0 and variance σ_0^2.

(a) Show that for $t > 0$ we can write

$$Y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \phi^3 e_{t-3} + \cdots + \phi^{t-1} e_1 + \phi^t Y_0.$$

(b) Show that for $t > 0$ we have $E(Y_t) = \phi^t \mu_0$.

(c) Show that for $t > 0$

$$Var(Y_t) = \begin{cases} \dfrac{1-\phi^{2t}}{1-\phi^2}\sigma_e^2 + \phi^{2t}\sigma_0^2 & \text{for } \phi \neq 1 \\ t\sigma_e^2 + \sigma_0^2 & \text{for } \phi = 1 \end{cases}$$

(d) Suppose now that $\mu_0 = 0$. Argue that, if $\{Y_t\}$ is stationary, we must have $\phi \neq 1$.

(e) Continuing to suppose that $\mu_0 = 0$, show that, if $\{Y_t\}$ is stationary, then $Var(Y_t) = \sigma_e^2/(1-\phi^2)$ and so we must have $|\phi| < 1$.

Appendix B: The Stationarity Region for an AR(2) Process

In the second-order case, the roots of the quadratic characteristic polynomial are easily found to be

$$\frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_2} \tag{4.B.1}$$

For stationarity we require that these roots exceed 1 in absolute value. We now show that this will be true if and only if three conditions are satisfied:

$$\phi_1 + \phi_2 < 1, \qquad \phi_2 - \phi_1 < 1, \qquad \text{and} \qquad |\phi_2| < 1 \tag{4.B.2}$$

Proof: Let the reciprocals of the roots be denoted G_1 and G_2. Then

$$G_1 = \frac{2\phi_2}{-\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}} = \frac{2\phi_2}{-\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}\left[\frac{-\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{-\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}\right]$$

$$= \frac{2\phi_2(-\phi_1 + \sqrt{\phi_1^2 + 4\phi_2})}{\phi_1^2 - (\phi_1^2 + 4\phi_2)} = \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2}$$

Similarly,

$$G_2 = \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2}$$

We now divide the proof into two cases corresponding to real and complex roots. The roots will be real if and only if $\phi_1^2 + 4\phi_2 \geq 0$.

I. **Real Roots**: $|G_i| < 1$ for $i = 1$ and 2 if and only if

$$-1 < \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2} < \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2} < 1$$

or

$$-2 < \phi_1 - \sqrt{\phi_1^2 + 4\phi_2} < \phi_1 + \sqrt{\phi_1^2 + 4\phi_2} < 2.$$

Consider just the first inequality. Now $-2 < \phi_1 - \sqrt{\phi_1^2 + 4\phi_2}$ if and only if $\sqrt{\phi_1^2 + 4\phi_2} < \phi_1 + 2$ if and only if $\phi_1^2 + 4\phi_2 < \phi_1^2 + 4\phi_1 + 4$ if and only if $\phi_2 < \phi_1 + 1$, or $\phi_2 - \phi_1 < 1$.

The inequality $\phi_1 + \sqrt{\phi_1^2 + 4\phi_2} < 2$ is treated similarly and leads to $\phi_2 + \phi_1 < 1$.

These equations together with $\phi_1^2 + 4\phi_2 \geq 0$ define the stationarity region for the real root case shown in Exhibit 4.17.

II. **Complex Roots**: Now $\phi_1^2 + 4\phi_2 < 0$. Here G_1 and G_2 will be complex conjugates and $|G_1| = |G_2| < 1$ if and only if $|G_1|^2 < 1$. But $|G_1|^2 = [\phi_1^2 + (-\phi_1^2 - 4\phi_2)]/4 = -\phi_2$ so that $\phi_2 > -1$. This together with the inequality $\phi_1^2 + 4\phi_2 < 0$ defines the part of the stationarity region for complex roots shown in Exhibit 4.17 and establishes Equation (4.3.11). This completes the proof.

Appendix C: The Autocorrelation Function for ARMA(*p*,*q*)

Let $\{Y_t\}$ be a stationary, invertible ARMA(p,q) process. Recall that we can always write such a process in general linear process form as

$$Y_t = \sum_{j=0}^{\infty} \psi_j e_{t-j} \tag{4.C.1}$$

where the ψ-weights can be obtained recursively from Equations (4.4.7), on page 79. We then have

$$E(Y_{t+k}e_t) = E\left(\sum_{j=0}^{\infty} \psi_j e_{t+k-j} e_t\right) = \psi_k \sigma_e^2 \text{ for } k \ge 0 \tag{4.C.2}$$

Thus the autocovariance must satisfy

$$\begin{aligned}\gamma_k &= E(Y_{t+k}Y_t) = E\left[\left(\sum_{j=1}^{p} \phi_j Y_{t+k-j} - \sum_{j=0}^{q} \theta_j e_{t+k-j}\right) Y_t\right] \\ &= \sum_{j=1}^{p} \phi_j \gamma_{k-j} - \sigma_e^2 \sum_{j=k}^{q} \theta_j \psi_{j-k}\end{aligned} \tag{4.C.3}$$

where $\theta_0 = -1$ and the last sum is absent if $k > q$. Setting $k = 0, 1, \ldots, p$ and using $\gamma_{-k} = \gamma_k$ leads to $p + 1$ linear equations in $\gamma_0, \gamma_1, \ldots, \gamma_p$.

$$\left.\begin{aligned}\gamma_0 &= \phi_1\gamma_1 + \phi_2\gamma_2 + \cdots + \phi_p\gamma_p - \sigma_e^2(\theta_0 + \theta_1\psi_1 + \cdots + \theta_q\psi_q) \\ \gamma_1 &= \phi_1\gamma_0 + \phi_2\gamma_1 + \cdots + \phi_p\gamma_{p-1} - \sigma_e^2(\theta_1 + \theta_2\psi_1 + \cdots + \theta_q\psi_{q-1}) \\ &\vdots \\ \gamma_p &= \phi_1\gamma_{p-1} + \phi_2\gamma_{p-2} + \cdots + \phi_p\gamma_0 - \sigma_e^2(\theta_p + \theta_{p+1}\psi_1 + \cdots + \theta_q\psi_{q-p})\end{aligned}\right\} \tag{4.C.4}$$

where $\theta_j = 0$ if $j > q$.

For a given set of parameter values σ_e^2, φ's, and θ's (and hence ψ's), we can solve the linear equations to obtain $\gamma_0, \gamma_1, \ldots, \gamma_p$. The values of γ_k for $k > p$ can then be evaluated from the recursion in Equations (4.4.8), on page 79. Finally, ρ_k is obtained from $\rho_k = \gamma_k/\gamma_0$.